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Research article

One Approach to Construction of Bilateral Approximations Methods for Solution of Nonlinear Eigenvalue Problems

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Abstract

This article continues the study of a new approach to construction the methods and algorithms of bilateral approximations to the eigenvalues of nonlinear spectral problems. Based on Newton's method and Halley's method the algorithms of alternating and including bilateral approximations to the eigenvalues are constructed and justified.

Keywords: nonlinear eigenvalue problem, numerical algorithm, derivatives of matrix determinant, bilateral approximations, alternating approximations, including approximations.

1. Introduction

A large number of works (see, eg, [1-7]) is devoted to Halley's method, the method of third order for solving nonlinear equation

$$f(\lambda) = 0 . \tag{1}$$

For real-valued function $f(\lambda)$ the method is written as

$$\lambda_{m+1} = \lambda_m - \frac{f(\lambda_m)}{f'(\lambda_m)\left(1 - \frac{f(\lambda_m) f''(\lambda_m)}{2 f'(\lambda_m)^2}\right)}, \quad m = 0,1,\dots \,. \tag{2}$$

This method is also called the method of tangent hyperbolas [8].

The most complete results about the method (2) can be obtained from [3, 6], where also a rather complete list of references is presented. Many authors, exploring Halley's method develop convergence theorems, in particular convergence theorems similar to the so-called Newton-Kantorovich theorems (see eg. [9]). In this direction, the most complete results can be found in [3].

In particular, the Halley's method is also obtained within the framework of approach [10-14] to constructing iterative processes of bilateral approximations to the solution of nonlinear equations, which is described in [15]. The essence of this approach is to construct bilateral (the alternating or including) approximations to the solution using a combination of Newton's method for function $f(\lambda)$ and some auxiliary function. In this approach, a number of new properties of Halley's method , which are not detected and not investigated in any of the known to the author of the works, are obtained. Namely, for a certain class of functions with an appropriate choice of initial approximations to give a monotone approximation to the solution of a nonlinear equation both from the bottom and the top, starting with $m = 0$. Adaptation of the well-known theorems of convergence for Halley's method allows to obtain the conditions which separate the class of functions $f(\lambda)$ for which the iterative process (2) of Halley's method gives the alternated approximations to the solution of equation (1) and the class of functions for which we can construct iterative processes that give a monotone including approximations.

2. Statement of the problem and some preliminary results

We consider the nonlinear eigenvalue problem

$$\mathbf{D}(\lambda) y = 0 \,,$$

where $\mathbf{D}(\lambda)$ is a square matrix of order n, all elements of which are sufficiently smooth (at least twice continuously differentiable) functions of the parameter $\lambda \in R$, $y \in R^n$. Eigenvalues are sought for solutions of the determinant equation

$$f(\lambda) \equiv \det \mathbf{D}(\lambda) = 0 \,. \tag{3}$$

To determine the isolated eigenvalue of matrix we propose and justify the Newton-type iterative processes that give the alternating approximations to the root of the equation (3), i.e

$$\lambda_0 < \lambda_2 < \dots < \lambda_{2m} < \dots < \lambda^* < \dots < \lambda_{2m-1} < \dots < \lambda_3 < \lambda_1$$

or

$$\lambda_1 < \lambda_3 < \dots < \lambda_{2m-1} < \dots < \lambda^* < \dots < \lambda_{2m} < \dots < \lambda_2 < \lambda_0$$

and the including monotonous bilateral approximations to the root, i.e.

$$\mu_0 < \mu_1 < \dots < \mu_{2m} < \dots < \lambda^* < \dots < \nu_{2m} < \dots < \nu_1 < \nu_0$$

without revealing in so doing the determinant $\det \mathbf{D}(\lambda)$. This means that the left hand side of equation (3) in explicit form is not set, but the algorithm of finding the functions $f(\lambda)$ and their derivatives $f'(\lambda)$ and $f''(\lambda)$ at a fixed value of parameter λ, using the *LU*-decomposition of the matrix $\mathbf{D}(\lambda)$ is proposed. This algorithm is based on the fact that the matrix $\mathbf{D}(\lambda)$ of order n, in which at any given value $\lambda=\lambda_m$ the principal minors of all orders from 1 to $(n-1)$ differ from zero, by *LU* - decomposition can be written as

$$\mathbf{D}(\lambda)=\mathbf{L}(\lambda)\mathbf{U}(\lambda)\ ,$$

where $\mathbf{L}(\lambda)$ is the lower triangular matrix with single diagonal elements, and $\mathbf{U}(\lambda)$ is the upper triangular matrix. Then

$$f(\lambda)=\det \mathbf{L}(\lambda)\det \mathbf{U}(\lambda)=\prod_{i=1}^{n} u_{ii}(\lambda)$$

Since the elements of a square matrix $\mathbf{D}(\lambda)$ (and, therefore, the matrix $\mathbf{U}(\lambda)$) are differentiable function, with respect to λ, then for any λ we obtain that

$$f'(\lambda)=\sum_{k=1}^{n} v_{kk}(\lambda)\prod_{i=1,\ i\neq k}^{n} u_{ii}(\lambda)\ ,$$

$$f''(\lambda)=\sum_{k=1}^{n} w_{kk}(\lambda)\prod_{i=1,\ i\neq k}^{n} u_{ii}(\lambda)++\sum_{k=1}^{n} v_{kk}(\lambda)\left(\sum_{j=1,\ j\neq k}^{n} v_{jj}(\lambda)\prod_{i=1,\ i\neq k,\ i\neq j}^{n} u_{ii}(\lambda)\right)$$

where $v_{ii}(\lambda)=u'_{ii}(\lambda)$ and $w_{ii}(\lambda)=v'_{ii}(\lambda)$ are the elements of matrices $\mathbf{V}(\lambda)$ and $\mathbf{W}(\lambda)$ in such decompositions

$$\mathbf{D}'(\lambda)\equiv\mathbf{B}(\lambda)=\mathbf{M}(\lambda)\mathbf{U}(\lambda)+\mathbf{L}(\lambda)\mathbf{V}(\lambda)\ ,$$

$$\mathbf{D}''(\lambda)\equiv\mathbf{C}(\lambda)=\mathbf{N}(\lambda)\mathbf{U}(\lambda)+2\mathbf{M}(\lambda)\mathbf{V}(\lambda)+\mathbf{L}(\lambda)\mathbf{W}(\lambda)\,.$$

Therefore, to calculate $f(\lambda_m)$, $f'(\lambda_m)$ and $f''(\lambda_m)$ needed, for a fixed $\lambda=\lambda_m$, calculate

$$\begin{aligned}&\mathbf{D}=\mathbf{LU}\,,\\&\mathbf{B}=\mathbf{MU}+\mathbf{LV}\,,\\&\mathbf{C}=\mathbf{NU}+2\mathbf{MV}+\mathbf{LW}\,,\end{aligned}\qquad(4)$$

whence we obtain

$$f(\lambda_m)=\prod_{i=1}^{n} u_{ii}\ ,\quad f'(\lambda_m)=\sum_{k=1}^{n} v_{kk}\prod_{i=1,\ i\neq k}^{n} u_{ii}\ ,$$

$$f''(\lambda_m)=\sum_{k=1}^{n} w_{kk}\prod_{i=1,\ i\neq k}^{n} u_{ii}+\sum_{k=1}^{n} v_{kk}\left(\sum_{j=1,\ j\neq k}^{n} v_{jj}\prod_{i=1,\ i\neq k,\ i\neq j}^{n} u_{ii}\right).\qquad(5)$$

The elements of matrices in the decompositions (4) can be calculated using the corresponding recurrent relations written out in [14] (see also [12, 16, 17]).

So, not knowing the explicit dependence $f(\lambda)$ on λ, for any fixed λ we can find the value of $f(\lambda)$ and its derivatives. Therefore, for solving (3) we can use the methods that apply the derivatives, in particular, to construct the Newton-type methods, which give the bilateral approximation to the solution. This requires further study of function $f(\lambda)$, what is realized later in the work.

3. Iterative Halley's function and some of its properties

Along with three times continuously differentiable function $f(\lambda)$ of real variable that describes the nonlinear equation (1), consider the function

$$h(\lambda) = \frac{f(\lambda)}{\operatorname{sign} f'(\lambda)\sqrt{|f'(\lambda)|}},$$

which obviously has the same zeros as the function $f(\lambda)$. Easy to see that $h(\lambda)$ is twice continuously differentiable and at the point of λ^* the relations

$$h'(\lambda^*) = \sqrt{|f'(\lambda^*)|}, \qquad h''(\lambda^*) = 0. \tag{6}$$

are satisfied. Indeed since

$$h'(\lambda) = \sqrt{|f'(\lambda)|} - h(\lambda)\left(\frac{f''(\lambda)}{2f'(\lambda)}\right), \tag{7}$$

$$h''(\lambda) = h(\lambda)\left[\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)^2 - \left(\frac{f''(\lambda)}{2f'(\lambda)}\right)'\right], \tag{8}$$

then from (7) and (8) immediately follow the relations (6).

Now, using Newton's method to function $h(\lambda)$, we obtain Halley's method (2) for the function $f(\lambda)$

$$\lambda_{m+1} = \lambda_m - \frac{h(\lambda_m)}{h'(\lambda_m)}, \qquad m = 0, 1, \dots . \tag{9}$$

Thus, the properties of Halley's method are determined by the properties of function $h(\lambda)$. So now examine the properties of functions $h(\lambda)$ depending on the properties of the function $f(\lambda)$ and its derivatives. To do this, we consider, as in [15], four possible cases of behavior of function $f(\lambda)$ in some neighborhood U of a simple root λ^*.

(A). Function $f(\lambda)$ is convex ($f''(\lambda) > 0$) and its derivative is $f'(\lambda) < 0$.

(B). Function $f(\lambda)$ is concave ($f''(\lambda) < 0$) and its derivative is $f'(\lambda) < 0$.

(C). Function $f(\lambda)$ is convex ($f''(\lambda) > 0$) and its derivative is $f'(\lambda) > 0$.

(D). Function $f(\lambda)$ is concave ($f''(\lambda) < 0$) and its derivative is $f'(\lambda) > 0$.

Further we investigate the properties of function $h(\lambda)$ and its behavior.

Let $f(\lambda)$ be a decreasing and convex with respect to λ on U function, that is, $f'(\lambda)<0$ and $f''(\lambda)>0$ (the case (A)).

Since the function

$$s(\lambda)=\frac{f(\lambda)f''(\lambda)}{(f'(\lambda))^2}$$

at the point $\lambda=\lambda^*$ is equal to zero, then because of continuity of $s(\lambda)$ there is such neighborhood of the root

$$U_\varepsilon(\lambda^*)=\left\{\lambda:\left|\lambda-\lambda^*\right|<\varepsilon\right\},$$

in which

$$|s(\lambda)|=\left|\frac{f(\lambda)f''(\lambda)}{(f'(\lambda))^2}\right|\le q<1.$$

Since

$$h'(\lambda)=\sqrt{|f'(\lambda)|}-h(\lambda)\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)=\sqrt{|f'(\lambda)|}\left(1-\frac{f(\lambda)f''(\lambda)}{2f'(\lambda)^2}\right),$$

then it follows that in the neighborhood $U_\varepsilon(\lambda^*)$ the function $h'(\lambda)>0$. Now from the mean value theorem, applied to differentiable functions $h(\lambda)$ on the interval $[\mu,\lambda]\in U_\varepsilon(\lambda^*)$ we obtain

$$z(\lambda)-z(\mu)=z'(\xi)(\lambda-\mu)\ , \qquad \xi\in[\mu,\lambda]\ ,$$

whence it follows that the function $h(\lambda)$ is an increasing one.

Since $h(\lambda)$ is a twice continuously differentiable function, so the mean value theorem for derivative $h'(\lambda)>0$ on any segment that belongs in the neighborhood $[\mu,\lambda]\in U_\varepsilon(\lambda^*)$, i.e.

$$h'(\lambda)-h'(\mu)=h''(\xi)(\lambda-\mu),\quad \xi\in[\mu,\lambda]$$

will be satisfied. Thus, if $h''(\xi)<0$ the function $h'(\lambda)$ is decreasing function and it is increasing function if $h''(\xi)>0$.

To investigate the $h''(\lambda)$, we use the relation (8), i.e.

$$h''(\lambda)=h(\lambda)\cdot\beta(\lambda),$$

where

$$\beta(\lambda)=\left[\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)^2-\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)'\right].$$

We will show that $\beta(\lambda)$ is not identical to zero. Assume the opposite and make the replacement

$$\frac{f''(\lambda)}{2f'(\lambda)}=y(\lambda),$$

then we obtain the equation

$$y(\lambda)^2-y'(\lambda)=0.$$

This is a particular case of Riccati equation. Replacing $u(\lambda)=y^{-1}(\lambda)$ we reduce it to the Bernoulli equation

$$u'(\lambda)+1=0,$$

the solution of which is

$$u(\lambda) = -\lambda + C_1, \quad C_1 = \text{Const}.$$

Thus, we come to the equation

$$-(\lambda + C_1) f''(\lambda) = 2 f'(\lambda),$$

the solution of which is a function

$$f(\lambda) = -\frac{C_2}{\lambda + C_1} + C_3, \qquad C_2, C_3 - \text{Const}. \tag{10}$$

Thus, only for such $f(\lambda)$ $\beta(\lambda) \equiv 0$, but since $f(\lambda)$ appearance of (10) is not the subject of this work, that is we obtained that $\beta(\lambda)$ is not identical to zero for the class of functions under consideration.

Thus, if $\beta(\lambda) < 0$, then in a neighborhood of $U_\varepsilon \equiv U_\varepsilon(\lambda^*)$ the derivative $h'(\lambda)$ to the left hand of the root is an increasing function, and $h(\lambda)$ is a convex one, and to the right of the root $h'(\lambda)$ is a decreasing function and $h(\lambda)$ is a concave function. If $\beta(\lambda) > 0$, then in the neighborhood of $U_\varepsilon \equiv U_\varepsilon(\lambda^*)$ to the left hand of the root $h'(\lambda)$ is a decreasing function and $h(\lambda)$ is a concave one, and to the right hand of the root $h'(\lambda)$ is an increasing function and $h(\lambda)$ is a convex one.

Analogously, determine the properties for the function $h(\lambda)$ for the cases (B), (C) and (D), which are formulated in the following statement.

Theorem 1. *Let* λ^* *be a simple real root of equation* (1) *in some neighborhood of* U *of which for the function* $f(\lambda)$ *one of the conditions* (A) - (D) *is satisfied. Then there is a neighborhood of the root* $U_\varepsilon \subset U$, *in which:*

(T1) *when the condition*

$$\beta(\lambda) = \left[\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)^2 - \left(\frac{f''(\lambda)}{2f'(\lambda)}\right)'\right] < 0 \tag{11}$$

is satisfied, the function $h(\lambda) = f(\lambda)/\text{sign}\, f'(\lambda)\sqrt{|f'(\lambda)|}$ to the left of the root *is a convex monotonically increasing function, its derivative* $h'(\lambda) > 0$ *and it increases monotonically, and to the right of the root the function* $h(\lambda)$ *is a concave monotonically increasing function, its derivative* $h'(\lambda) > 0$ *and it decreases monotonically;*

(T2) *when the condition*

$$\beta(\lambda) = \left[\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)^2 - \left(\frac{f''(\lambda)}{2f'(\lambda)}\right)'\right] > 0 \tag{12}$$

is satisfied, the function $h(\lambda) = f(\lambda)/\text{sign}\, f'(\lambda)\sqrt{|f'(\lambda)|}$ *to the left of the root is a concave monotonically increasing function, its derivative* $h'(\lambda) > 0$ *and it monotonically decreases, and to the right of the root the function* $h(\lambda)$ *is a convex monotonically increasing function, its derivative* $h'(\lambda) > 0$ *and it monotonically increases* .

Thus, Theorem 1 determines the properties of function $h(\lambda)$, and **Figure 1** and **Figure 2** illustrate its behavior depending on the properties of function $f(\lambda)$ in some neighborhood of the root λ^*. **Figure 1** and **Figure 2** correspond to the conclusion (T1) and (T2) of the theorem, accordingly.

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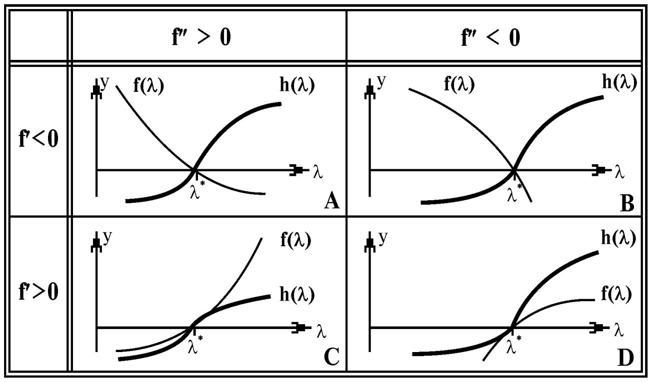

Figure 1: Behavior of the functions $f(\lambda)$ and $h(\lambda)$ in the neighborhood of a simple real root λ^* of functions $f(\lambda)$ at $\beta(\lambda) < 0$.

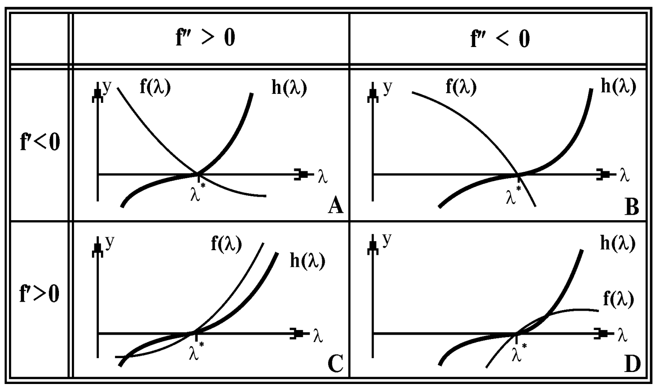

Figure 2: Behavior of the functions $f(\lambda)$ and $h(\lambda)$ in the neighborhood of a simple real root λ^* of functions $f(\lambda)$ at $\beta(\lambda) > 0$.

4. Monotone approximations

Such behavior of the function $h(x)$ allows from the iterative formula (9), and hence (2), to obtain a monotonous sequence of approximations to the root. So Halley's method has the following monotone properties.

Theorem 2. *If in some neighborhood of a simple root* λ^* *of the equation* (1)

$$U_\varepsilon(\lambda^*) = \{\ \lambda : |\ \lambda - \lambda^*\ | < \varepsilon\ \},$$

in which

$$\left| \frac{f(\lambda) f''(\lambda)}{f'(\lambda)^2} \right| \le q < 1,$$

for three times continuously differentiable function $f(\lambda)$, *describing the equation* (1), *one of the conditions* (A) - (D), *condition* (11), *and condition*

$$\max_{\lambda \in U_\delta(\lambda^*)} \left| \frac{h(\lambda) h''(\lambda)}{h'(\lambda)^2} \right| < 1, \tag{13}$$

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are satisfies, where $h(\lambda) = f(\lambda)/\text{sign}\, f'(\lambda)\sqrt{|f'(\lambda)|}$*, then even numbers of sequence* $\{\lambda_m\}$*, obtained by Halley's method* (2)*, form a monotonically increasing sequence, and odd numbers form a monotonically descending sequence if* $\lambda_0 \in (\lambda^* - \varepsilon, \lambda^*)$*, and conversely, if* $\lambda_0 \in (\lambda^*, \lambda^* + \varepsilon)$.

Theorem 3. *If in some neighborhood of a simple root* λ^* *of the equation* (1)

$$U_\varepsilon(\lambda^*) = \{\ \lambda : |\ \lambda - \lambda^*\ | < \varepsilon\ \},$$

in which

$$\left|\frac{f(\lambda) f''(\lambda)}{f'(\lambda)^2}\right| \le q < 1$$

for three times continuously differentiable function $f(\lambda)$ *describing equation* (1)*, one of the conditions* (A) - (D)*, and condition* (12)*, are satisfies, then, begining from m=0, the sequence* $\{\lambda_m\}$*, obtained by Halley's method* (2) *increases monotonically if* $\lambda_0 \in (\lambda^* - \varepsilon, \lambda^*)$*, and monotonically decreases if* $\lambda_0 \in (\lambda^*, \lambda^* + \varepsilon)$.

Thus, from the Theorems 1, 2 and 3, it follows that Halley's method (2) for a certain class of functions, which describe the nonlinear equation gives the alternating approximation to the solution of this equation on both sides, and for another class of functions - only unilateral monotone approximation unlike Newton's method the monotone approximations begins with $m = 0$.

5. Constructing iterative process including bilateral approximations and justification of convergence

Consider the class of functions $f(\lambda)$, for which the condition (12) is satisfied, i.e. the class of functions for which Halley’s method (2) gives only one-sided monotone approximation.

To construct a sequence of bilateral approximations we use the properties of the function $h(\lambda)$ and the iterative process is written as

$$\begin{cases} \mu_{m+1} = \mu_m - \dfrac{f(\mu_m)}{f'(\mu_m)\left(1 - \dfrac{f(\mu_m) f''(\mu_m)}{2 f'(\mu_m)^2}\right)}, \\[2ex] \nu_{m+1} = \nu_m - \dfrac{f(\nu_m)}{f'(\nu_m)\left(1 - \dfrac{f(\nu_m) f''(\nu_m)}{2 f'(\nu_m)^2}\right)}, \end{cases} \quad m = 0, 1, \ldots, \ \mu_0 \in \left(\lambda^* - \varepsilon, \lambda^*\right), \quad \nu_0 \in \left(\lambda^*, \lambda^* + \varepsilon\right). \tag{14}$$

Bilateral convergence of the iterative process (14) for the function $f(\lambda)$ is shown in **Figure 3**, and its justification is a theorem.

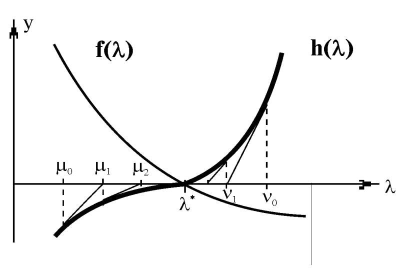

Figure 3

Theorem 4. *Let* λ^* *be a simple real root of the equation* (1) *and let in some neighborhood of the root*

$$U_\varepsilon(\lambda^*) = \{\ \lambda : |\ \lambda - \lambda^*\ | < \varepsilon\ \},$$

in which

$$\left| \frac{f(\lambda) f''(\lambda)}{f'(\lambda)^2} \right| \le q < 1$$

for three times continuously differentiable function $f(\lambda)$ *describing equation* (1), *one of the conditions* (A) - (D) *and condition* (12) *are satisfied, i.e.*

$$\left[\left(\frac{f''(\lambda)}{2 f'(\lambda)} \right)^2 - \left(\frac{f''(\lambda)}{2 f'(\lambda)} \right)' \right] > 0 .$$

Also, let for the function $h(\lambda) = f(\lambda) / \text{sign}\, f'(\lambda) \sqrt{|\ f'(\lambda)\ |}$

$$m_2 = \min_{\lambda \in U_\delta(\lambda^*)} |h'(\lambda)|, \qquad M_2 = \max_{\lambda \in U_\delta(\lambda^*)} |h''(\lambda)|,$$

moreover

$$\frac{M_2}{2m_2} |\ \mu_0 - \lambda^*\ | < 1 \qquad \frac{M_2}{2m_2} |\ \nu_0 - \lambda^*\ | < 1 .$$

Then, if $\mu_0 \in (\lambda^* - \varepsilon, \lambda^*)$ *and* $\nu_0 \in (\lambda^*, \lambda^* + \varepsilon)$, *then the iterative process* (14) *converges to the solution of* λ^* *on both sides*

$$\mu_0 < \mu_1 < \ldots < \mu_m < \ldots < \lambda^* < \ldots < \nu_m < \ldots < \nu_1 < \nu_0 ,$$

moreover for the error from the left and right the evaluations

$$|\mu_m - \lambda^*| < q_\mu^{2^m - 1} |\mu_0 - \lambda^*|, \quad |\nu_m - \lambda^*| < q_\nu^{2^m - 1} |\nu_0 - \lambda^*|$$

hold true, where

$$q_\mu = \frac{M_2}{2m_2} |\mu_0 - \lambda^*| < 1, \qquad q_\nu = \frac{M_2}{2m_2} |\nu_0 - \lambda^*| < 1.$$

If

$$|h'(\lambda)| < N, \qquad \text{and} \qquad \left| \left(\frac{f''(\lambda)}{2 f'(\lambda)} \right)^2 - \left(\frac{f''(\lambda)}{2 f'(\lambda)} \right)' \right| < M ,$$

moreover

$$\left(\frac{NM}{2m_2} \right)^{\frac{1}{2}} |\mu_0 - \lambda^*| < 1, \qquad \left(\frac{NM}{2m_2} \right)^{\frac{1}{2}} |\nu_0 - \lambda^*| < 1,$$

then for the error from the left the estimations

$$|\mu_{m+1} - \lambda^*| \le \frac{NM}{2m_2} |\mu_m - \lambda^*|^3, \quad |\mu_m - \lambda^*| < q_\mu^{3^m - 1} |\mu_0 - \lambda^*|, \quad \text{where} \quad q_\mu = \left(\frac{NM}{2m_2} \right)^{\frac{1}{2}} |\mu_0 - \lambda^*| < 1$$

hold true, and for the error from the right the estimations

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$$|\nu_{m+1}-\lambda^*| \le \frac{NM}{2m_2}|\nu_m-\lambda^*|^3, |\nu_m-\lambda^*| < q_\nu^{3^m-1}|\nu_0-\lambda^*|, \quad where \quad q_\nu = \left(\frac{NM}{2m_2}\right)^{\frac{1}{2}} |\nu_0-\lambda^*| < 1$$

hold true.

Remark 1. Iterative process including approximations can be constructed in a different form, but you need to know the behavior of the function $f(\lambda)$. For example, if we have two initial approximations μ_0 and ν_0 from the left hand and right hand of the root, respectively, and the behavior of the function $f(\lambda)$ is the responsible for the assumption (A), then, considering the properties of functions $f(\lambda)$ and $h(\lambda)$, an iterative process can be constructed as

$$\begin{cases} \mu_{m+1} = \mu_m - \dfrac{f(\mu_m)}{f'(\mu_m)}, \\ \nu_{m+1} = \nu_m - \dfrac{h(\nu_m)}{h'(\nu_m)}, \end{cases} \qquad m = 0,1, \ldots .$$

Including monotone approximations of the iterative process shown in **Figure 4** and for it valid analogue of Theorem 4, where the estimates for the errors to the left of the root will be known estimates of error of Newton's method.

Remark 2. Including approximation can be constructed having one initial approximation. For example, if we have an initial approximation to the right of the root, and the behavior of the function $f(\lambda)$ again corresponds to the assumption (A), then the iterative process can be constructed as

$$\begin{cases} \mu_{m+1} = \nu_m - \dfrac{f(\nu_m)}{f'(\nu_m)}, \\ \nu_{m+1} = \nu_m - \dfrac{h(\nu_m)}{h'(\nu_m)}, \end{cases} \qquad m = 0,1, \ldots .$$

The behavior of the approximations of the iterative process illustrated in **Figure 5** and for it also exists analog of Theorem 4 except the estimates of errors to the left of the root.To test the proposed iterative processes consider the model eigenvalue problem with quadratic dependence on the parameter.

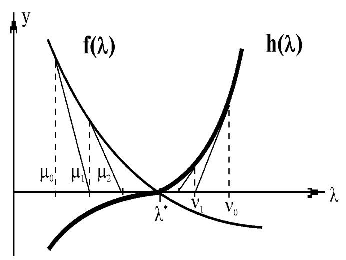

Figure 4

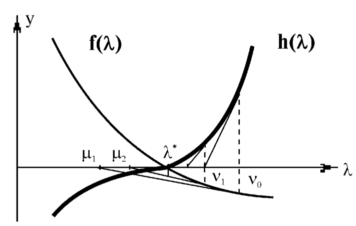

Figure 5

6. The alternating bilateral convergence of the Halley's method and its justification

Consider the class of functions $f(\lambda)$ for which the condition (11) is satisfied, and apply for solving equation (1) that is described by the functions, the Halley's method (2).

We show that approximations to the root of the equation are carried out alternately on both sides.

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Indeed, bilateral convergence of the iterative process (9) (and hence equivalent thereto iterative process (2) for the function $f(\lambda)$) is shown in **Figure 6**, and its justification is a theorem.

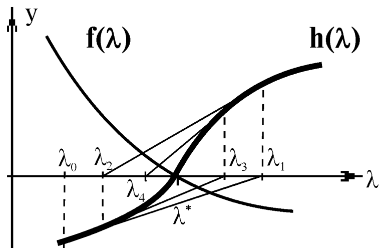

Figure 6

Theorem 5. *Let* λ^* *be a simple real root of the equation* (1) *and let in some neighborhood of the root*

$$U_\varepsilon(\lambda^*) = \{\ \lambda : |\ \lambda - \lambda^*\ | < \varepsilon\ \},$$

in which

$$\left|\frac{f(\lambda)\,f''(\lambda)}{f'(\lambda)^2}\right| \le q < 1$$

for three times continuously differentiable function $f(\lambda)$ *describing equation* (1), *one of the conditions* (A) - (D) *and condition* (11) *be satisfied, i.e.*

$$\left[\left(\frac{f''(\lambda)}{2\,f'(\lambda)}\right)^2 - \left(\frac{f''(\lambda)}{2\,f'(\lambda)}\right)'\right] < 0\,.$$

Also, let the function $h(\lambda) = f(\lambda)/\mathrm{sign}\, f'(\lambda)\sqrt{|\ f'(\lambda)\,|}$

$$m_2 = \min_{\lambda\in U_\delta(\lambda^*)} |h'(\lambda)|\,, \qquad M_2 = \max_{\lambda\in U_\delta(\lambda^*)} |h''(\lambda)|\,,$$

moreover

$$\frac{M_2}{2m_2}\left|\lambda_0 - \lambda^*\right| < 1\,, \qquad \frac{M_2}{2m_2}\left|\lambda_1 - \lambda^*\right| < 1$$

and the condition (13) *be satisfied, namely:*

$$\max_{\lambda\in U_\delta(\lambda^*)}\left|\frac{h(\lambda)h''(\lambda)}{h'(\lambda)^2}\right| < 1\,.$$

Then, if $\lambda_0 \in \left(\lambda^* - \varepsilon, \lambda^*\right)$ *and* $\left(\lambda_0 \in \left(\lambda^*, \lambda^* + \varepsilon\right)\right)$, *the iterative process* (2) *converges to solution of* λ^* *on both sides*

$$\lambda_0 < \lambda_2 < \ldots < \lambda_{2m} < \ldots < \lambda^* < \ldots < \lambda_{2m-1} < \ldots < \lambda_3 < \lambda_1$$
$$\left(\lambda_1 < \lambda_3 < \ldots < \lambda_{2m-1} < \ldots < \lambda^* < \ldots < \lambda_{2m} < \ldots < \lambda_2 < \lambda_0\right),$$

moreover for the error from the left and right evaluations, are true respectively

$$\left|\lambda_{2m}-\lambda^*\right|<q_0^{4^m-1}\left|\lambda_0-\lambda^*\right|,\ \left|\lambda_{2m-1}-\lambda^*\right|<q_1^{4^m-1}\left|\lambda_1-\lambda^*\right|,$$

where

$$q_0=\frac{M_2}{2m_2}\left|\lambda_0-\lambda^*\right|<1,\qquad q_1=\frac{M_2}{2m_2}\left|\lambda_1-\lambda^*\right|<1.$$

If

$$\left|h'(\lambda)\right|<N,\qquad a\qquad \left|\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)^2-\left(\frac{f''(\lambda)}{2f'(\lambda)}\right)'\right|<M,$$

moreove

$$\left(\frac{NM}{2m_2}\right)^{\frac{1}{2}}\left|\lambda_0-\lambda^*\right|<1,\qquad \left(\frac{NM}{2m_2}\right)^{\frac{1}{2}}\left|\lambda_1-\lambda^*\right|<1,$$

then for the error from the left the estimations are valid

$$\left|\lambda_{2m}-\lambda^*\right|<q_0^{9^m-1}\left|\lambda_0-\lambda^*\right|,\ \ q_0=\left(\frac{NM}{2m_2}\right)^{\frac{1}{2}}\left|\lambda_0-\lambda^*\right|<1,$$

and for the error from the right the estimations

$$\left|\lambda_{2m-1}-\lambda^*\right|<q_1^{9^m-1}\left|\lambda_1-\lambda^*\right|,\ \ ,q_1=\left(\frac{NM}{2m_2}\right)^{\frac{1}{2}}\left|\lambda_1-\lambda^*\right|<1$$

hold true.

Remark 3. From a practical point of view research of the sign of function $\beta(\lambda)$ is not a simple task. But belonging of function $f(\lambda)$ to a particular class can be determined by the first two iterations the method (9) (in fact, from the analysis of behavior of the sign of correction $h(\lambda)/h'(\lambda)$) and then using the behavior of functions $f(\lambda)$ and $h(\lambda)$ (Theorem 1), to correct if necessary the iterative process so as to obtain bilateral approximations. For this purpose the iterative process can be written, for example, as (if $\nu_0\in\left(\lambda^*,\lambda^*+\varepsilon\right)$)

$$\begin{cases}\mu_{m+1}=\nu_m-\dfrac{f(\nu_m)}{f'(\nu_m)},\\ \nu_{m+1}=\nu_m-\dfrac{h(\nu_m)}{h'(\nu_m)},\end{cases}\qquad m=0,1,\ \ldots,\tag{15}$$

twat does not give the same rate of convergence from different sides of the root, or (15) to use only for obtain the first approximation on the other side of the root, and then to continue the iterative process by the formula (14) There may also be other options.

7. Algorithmic realization of iterative processes

To test the proposed iterative processes we consider the model eigenvalue problem with exponential dependence on the parameter [14]

$$\mathbf{D}(\lambda)=\mathbf{A}+\lambda\mathbf{E}+e^{-\lambda}\mathbf{E},\tag{16}$$

where $\mathbf{E}$ is the identity $n\times n$ matrix, $\mathbf{A}$ is the three-diagonal $n\times n$ matrix with nonzero elements $a_{i-1,i}=a_{i,i+1}=1,\ a_{ii}=-2$.

The eigenvalues will be found as solutions of the determinant equation

$$f(\lambda) \equiv \det \mathbf{D}(\lambda) = 0 \,. \tag{17}$$

For this we use Halley's iterative process (9), which we rewrite in the equivalent form

$$\lambda_{m+1} = \lambda_m - \frac{2 f(\lambda_m) f'(\lambda_m)}{2 f'(\lambda_m)^2 - f(\lambda_m) f''(\lambda_m)}, \; m = 0,1,\dots \; . \tag{18}$$

If to replaced the value of the function and its derivatives at the points λ_m by the relations (5), obtained from the decomposition (4), then the iterative process (18) takes the form

$$\lambda_{m+1} = \lambda_m - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}} \right) / \sum_{k=1}^{n} \left(\left(\frac{v_{kk}}{u_{kk}} \right)^2 - \frac{1}{2} \frac{w_{kk}}{u_{kk}} + \frac{1}{2} \frac{v_{kk}}{u_{kk}} \left(\sum_{i=1,\, i \neq k}^{n} \frac{v_{ii}}{u_{ii}} \right) \right) \qquad m = 0,1,\,\dots, \tag{19}$$

where u_{kk} , v_{kk} , w_{kk} are the elements of matrix $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{W}$ in the expansion (4) for fixed $\lambda = \lambda_m$.

Thus, the algorithm can be written in the following.

Algorithm 1.

Step 1. Set the initial approximation λ_0 to the *s*-th eigenvalue of the problem (17).

Step 2. **for** *m*=0,1,2, … until convergence **do**

Step 3. Compute the values u_{kk} , v_{kk} , w_{kk} from the decompositions (4) at $\lambda = \lambda_m$.

Step 4. Compute the approximation to the eigenvalue λ_{2m+2} by (19).

Step 5. **end for** *m*

We apply this algorithm to the problem (16) and compare the calculation results with the results obtained in [14]. Calculations were carried out with precision $\varepsilon = 10^{-6}$. Results of numerical experiments for two eigenvalue are given in **Table 1**

Table 1. Successive approximations to the eigenvalues of the problem (17)

λ_0	m	Algorithm 1	Algorithm with [14]
		λ_m	λ_m
4.0	1	3.915021275	3.941742455
	2	3.898891876	3.885413100
	3	3.898718071	3.900571617
	4	3.898718062	3.898687797
	5	-	3.898718074
	6	-	3.898718063
3.4	1	3.297936152	3.324599471
	2	3.271923048	3.271814
	3	3.271782747	3.271783
	4	3.271782746	3.272783

The table shows that the Algorithm 1 provides only one-sided approximation. Modify Algorithm 1 so as to obtain bilateral, but already including approximations. This can be done in at least by two ways.

Method one. According to Remark 2 the iterative process can be represented in the form (15), provided that the initial approximation is to the right of the eigenvalue and function satisfies the condition (A). There are other options. Thus, in order to apply a version of the iterative process, in a general case one needs to know the behavior of the function f and which side of the eigenvalue is chosen initial approximation. This is easily done algorithmically using the matrix elements of the decomposition (4) for computing f and f'', and determining their sings. So, if f and f'' have the opposite sings, the iterative process is written as

$$\begin{cases} \mu_{m+1} = \lambda_m - 1 / \sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}}, \\ \lambda_{m+1} = \lambda_m - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}} \right) / \sum_{k=1}^{n} \left[\left(\frac{v_{kk}}{u_{kk}} \right)^2 - \frac{1}{2} \frac{w_{kk}}{u_{kk}} + \frac{1}{2} \frac{v_{kk}}{u_{kk}} \left(\sum_{i=1,\, i \neq k}^{n} \frac{v_{ii}}{u_{ii}} \right) \right], \end{cases} \quad m = 0, 1, \ldots, \tag{20}$$

If the f and f'' have the same sings, the iterative process is written as

$$\begin{cases} \mu_{m+1} = \lambda_m - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}} \right) / \sum_{k=1}^{n} \left(\left(\frac{v_{kk}}{u_{kk}} \right)^2 - \frac{w_{kk}}{u_{kk}} \right), \\ \lambda_{m+1} = \lambda_m - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}} \right) / \sum_{k=1}^{n} \left[\left(\frac{v_{kk}}{u_{kk}} \right)^2 - \frac{1}{2} \frac{w_{kk}}{u_{kk}} + \frac{1}{2} \frac{v_{kk}}{u_{kk}} \left(\sum_{i=1,\, i \neq k}^{n} \frac{v_{ii}}{u_{ii}} \right) \right], \end{cases} \quad m = 0, 1, \ldots, \tag{21}$$

where u_{kk}, v_{kk}, w_{kk} are the elements of matrix $\mathbf{U}, \mathbf{V}$ and $\mathbf{W}$ in the expansion (4) for fixed $\lambda = \lambda_m$.

Thus, the algorithm is constructed as follows.

Algorithm 2.

Step 1. Set the initial approximation λ_0 to the *s*-th eigenvalue of the problem (17).

Step 2. **for** m=0,1,2, … until convergence **do**

Step 3. Compute the values u_{kk}, v_{kk}, w_{kk} from the decompositions (4) at $\lambda = \lambda_m$.

Step 4. Compute f and f'' by the formula (5).

Step 5. **if** ($f * f'' > 0$) **go to** Step 8.

Step 6. Compute the approximation to the eigenvalues μ_{m+1} and λ_{m+1} by (20).

Step 7. **go to** Step 9.

Step 8. Compute the approximation to the eigenvalues μ_{m+1} and λ_{m+1} by (21).

Step 9. **end for** *m*.

As a result of work of the Algorithm 2 we obtain the including approximations to the eigenvalue in the form

$$\mu_1 < \ldots < \mu_m < \ldots < \lambda_* < \ldots < \lambda_m < \ldots < \lambda_1 < \lambda_0 \ ,$$

if the initial approximation is to the right of the root, or in the form

$$\lambda_0 < \lambda_1 < \ldots < \lambda_m < \ldots < \lambda_* < \ldots < \mu_m < \ldots < \mu_1 ,$$

if the initial approximation is to the left of the root.

Second method. According to Section 5 the bilateral including approximations can be obtain by means of iteration process (14), but for this purpose it is needed to have two initial approximations to the left and to the right from an eigenvalue. If we have one initial approximation, then it is possible to use an iteration process (20) or (21) only to obtain the first approximation from two sides from an eigenvalue, and further to continue an iteration process by the formula (14). It gives an opportunity to obtain the cubical convergence to the eigenvalue from either side.

Thus, the iterative processes (20) and (21) play a supporting role and are used once, and all iterations are performed by the iterative process (14), which in equivalent record becomes

$$\begin{cases} \mu_{m+1} = \mu_m - \left(\sum_{k=1}^{n} \frac{v_{kk}}{u_{kk}} \right) / \sum_{k=1}^{n} \left[\left(\frac{v_{kk}}{u_{kk}} \right)^2 - \frac{1}{2}\frac{w_{kk}}{u_{kk}} + \frac{1}{2}\frac{v_{kk}}{u_{kk}} \left(\sum_{i=1,\, i \neq k}^{n} \frac{v_{ii}}{u_{ii}} \right) \right], \\ \nu_{m+1} = \nu_m - \left(\sum_{k=1}^{n} \frac{\bar{v}_{kk}}{\bar{u}_{kk}} \right) / \sum_{k=1}^{n} \left[\left(\frac{\bar{v}_{kk}}{\bar{u}_{kk}} \right)^2 - \frac{1}{2}\frac{\bar{w}_{kk}}{\bar{u}_{kk}} + \frac{1}{2}\frac{\bar{v}_{kk}}{\bar{u}_{kk}} \left(\sum_{i=1,\, i \neq k}^{n} \frac{\bar{v}_{ii}}{\bar{u}_{ii}} \right) \right], \end{cases} \quad m = 0, 1, \ldots, \tag{22}$$

where u_{kk} , v_{kk} , w_{kk} are the elements of matrix **U, V** and **W** in the expansion (4) for a fixed $\lambda = \mu_m$, and where $\bar{u}_{kk}$, $\bar{v}_{kk}$, $\bar{w}_{kk}$ are the elements of matrix **U, V** and **W** in the same expansion (4) for fixed $\lambda = \nu_m$.

Thus, the Algorithm will consist of the next steps.

Algorithm 3.

Step 1. Set the initial approximation λ_0 to the *s*-th eigenvalue of the problem (17).

Step 2. Compute the values u_{kk} , v_{kk} , w_{kk} from the decompositions (4) at $\lambda = \lambda_0$

Step 3. Compute f and f'' by the formula (5).

Step 4. **if** ($f * f'' > 0$) **go to** Step 7.

Step 5. Compute the approximation to the eigenvalues μ_1 and λ_1 by (20).

Step 6. **go to** Step 8.

Step 7. Compute the approximation to the eigenvalues μ_1 and λ_1 by (21).

Step 8. Put $\nu_1 = \lambda_1$

Step 9. **for** *m*=0,1,2, … until convergence **do**

Step 10. Compute the values u_{kk} , v_{kk} , w_{kk} from the decompositions (4) at $\lambda = \mu_m$.

Step 11. Compute the values $\bar{u}_{kk}$, $\bar{v}_{kk}$, $\bar{w}_{kk}$ from the decompositions (4) at $\lambda = \nu_m$.

Step 12. Compute the approximation to the eigenvalues μ_{m+1} and ν_{m+1} by (22).

Step 13. **end for** *m.*

The algorithm shows that for obtaining bilateral including approximations at each step one should turn twice to the algorithm for computing the decomposition (4). In this regard, it loses to Algorithm 2, but, as already mentioned, the Algorithm 3 has cubic rate of convergence of both sides to their eigenvalues as opposed to Algorithm 2, which has a cubic rate of convergence from one side only.

8. Conclusions

Approbation of constructed algorithms on model problems shows their reliability and efficiency, and also advantages in comparison with the usual Newton's method in the sense that at every step of iterative process we obtain two-sided estimates of the desired solution, and hence at each step we obtain comfortable a posteriori error estimates .

The proposed algorithms can be applied to the linear eigenvalue problems with respect to the spectral parameter, moreover if it is compared with existing methods and algorithms for obtaining lower bounds of eigenvalues of self-adjoint spectral problems, then they have the same advantages as the algorithms of the paper [15], namely:

• do not require knowledge of the lower border of the following (assuming that the eigenvalues are arranged in ascending order) eigenvalue, as for algorithms that are based on including theorems;

• do not require the construction of a basic operator with known eigenvalues such that the difference between the self-adjoint operator of the original problem and the base one was a positive operator, and construction of finite-dimensional perturbation of basic operator, as for the method of intermediate problems and its various modifications;

• do not require that the operator inverse to the original was compact operator and its determination as for Fichera method and its modifications.

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